Non-Abelian $\alpha_s^3/(m_a r^2)$ heavy-quark-antiquark potential

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We calculate the leading mass-suppressed two-loop contribution, proportional to $\alpha_s^3/(m_q r^2)$, to the potential of a heavy-quark-antiquark system. This contribution originates in the non-Abelian nature of quantum chromodynamics (QCD) and has no analogue in quantum electrodynamics. For this purpose, we elaborate a technique based on the implementation of the threshold expansion within the effective theory of nonrelativistic QCD. We discuss the phenomenological implications of our result for heavy-quarkonium physics. We also confirm a previous result for the two-loop $\mathcal{O}(\alpha_s^2)$ correction to the static potential.

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Since the pioneering work by Susskind way back in 1977 [1], the potential between a heavy quark and its antiquark has attracted the attention of many authors [2-8]. The structure of the heavy-quark-antiquark potential is crucial both for understanding the fundamental properties of QCD, such as confinement, and for describing the rich phenomenology of heavy quarkonia. While the potential, of course, cannot be found in a closed form, it can be systematically computed as an expansion in the strong-coupling constant α_s and the inverse of the heavy-quark mass m_q or, equivalently, in the heavy-quark velocity v. The leading-order (LO) approximation is given by the simple generalization of the Coulomb potential of OED. The purely relativistic $\mathcal{O}(v^2)$ correction is given, up to the color factor, by the standard Breit Hamiltonian. The one-loop $\mathcal{O}(\alpha_s)$ correction has been known for a long time [2], while the two-loop $\mathcal{O}(\alpha_s^2)$ correction has been computed only recently [7,8]. The leading and subleading mass-suppressed one-loop $\mathcal{O}(v\alpha_s)$ and $\mathcal{O}(v^2\alpha_s)$ corrections have been found in Refs. [4-6]. In this Rapid Communication, we compute the two-loop leading mass-suppressed $\mathcal{O}(v\,\alpha_s^2)$ term.

Let us take a closer look at this specific class of corrections. Owing to the gluon self-coupling, the QCD potential differs from its Abelian counterpart in two principal respects, namely by the running of the coupling constant [1] and the presence of generic corrections non-analytic in v^2 . In general, terms non-analytic in v^2 can also appear in QED as a result of iterations of operators that vanish on shell, in connection with the Coulomb singularity, which brings in a factor of 1/v. One can avoid such terms by including off-shell operators in the Hamiltonian, which is the standard practice in QED bound-state calculations. The non-analytic terms that we consider here are a consequence of the non-Abelian structure of QCD, and they are not necessarily related to the Coulomb singularity. They first appear at one loop as the well-known non-Abelian $\alpha_s^2/(m_a r^2)$ potential [4]. In this Rapid Communication, we take the next step and calculate this potential at two loops. Our main result is given by Eqs. (5)–(7). In order to derive it, we elaborate a technique based on the effective theory of nonrelativistic QCD (NRQCD) [10] implemented with the threshold expansion offering the possibility to systematically expand Feynman diagrams near threshold [9]. We expect this approach to be very useful for solving a wide class of problems of nonrelativistic dynamics.

The perturbative QCD potential is of special interest, since it is the basic ingredient for the theory of heavy quarkonia. In fact, the study of nonrelativistic heavy-quarkantiquark systems [11] and its applications to bottomonium [12] and top-antitop [13] physics rely entirely on first principles of QCD. In principle, these systems allow for a modelindependent perturbative treatment. The nonperturbative effects [14] are well under control for the top-antitop system and, at least within the sum-rule approach, also for bottomonium. This makes heavy quark-antiquark systems an ideal laboratory to determine the fundamental parameters of QCD, such as α_s and m_a . Recently, essential progress has been made in the theoretical investigation of the nonrelativistic heavy-quark threshold dynamics based on the effectivetheory approach [10]. The analytical results for the main parameters of the nonrelativistic heavy-quark-antiquark system are now available through next-to-next-to-leading order (NNLO) in α_s and v [15–19], and they have been applied to bottomonium [18–20] and top-antitop [21] phenomenology. Some specific classes of next-to-next-to-leading order (N³LO) corrections have also been investigated [22–24]; for a brief review, see Ref. [25]. These corrections have turned out to be so sizeable that it appears to be indispensable to gain full control over this order, both with regard to phenomenological applications and in order to understand the structure and the peculiarities of the nonrelativistic expansion. This Rapid Communication provides a major step in this direction.

In the following, we briefly outline the main features of our method, which is based on the combination of the concepts of threshold expansion [9] and effective theory [10]. The effective theory is designed to separate the scales and to perform the expansion in v at the level of the Lagrangian. Let us recall that the dynamics of a nonrelativistic quarkantiquark pair involves four different regions and corresponding modes [9]: (i) the hard region (energy and three-momentum scale like $m_q v$); (ii) the soft region (energy and three-momentum scales like $m_q v$); while three-momentum scales like $m_q v$); and (iv) the ultrasoft region (energy and three-momentum scale like $m_q v$). The ultrasoft region is only

relevant for gluons. NRQCD [10] is obtained by integrating out the hard modes. Subsequently integrating out the soft modes and the potential gluons results in the effective theory of potential NRQCD (PNRQCD) [26], which contains potential quarks and ultrasoft gluons as active particles. The dynamics of a nonrelativistic quark-antiquark pair in PNRQCD is governed by the effective Schrödinger equation and by its multipole interactions with the ultrasoft gluons. The corrections from the modes integrated out are contained in the higher-dimensional operators of the nonrelativistic Hamiltonian, corresponding to an expansion in v, and in the Wilson coefficients, which are expanded in α_s . Both NRQCD and PNRQCD have specific Feynman rules, which can be used for the computation of a systematic perturbative expansion. However, this problem is complicated because the expansion of the Lagrangian corresponds to a particular subspace of the total phase space. Thus, in a perturbative calculation within the effective theory, one has to formally impose some restrictions on the allowed values of the virtual momenta. Explicitly separating the phase space introduces additional scales to the problem, such as momentum cutoffs, and makes the approach much less transparent. A much more efficient and elegant method is based on the expansion by regions [9,27], which is a systematical method to expand Feynman diagrams in any limit of momenta and masses. It consists of the following steps: (i) consider various regions of a loop momentum k and expand, in every region, the integrand in Taylor series with respect to the parameters that are considered small there; (ii) integrate the expanded integrand over the whole integration domain of the loop momenta; (iii) put to zero any scaleless integral. In step (ii), dimensional regularization, with $d=4-2\epsilon$ space-time dimensions, is used to handle the divergences. In the case of the threshold expansion in v, one has to deal with the four regions and their scaling rules enumerated above.

In principle, the threshold expansion has to be applied to the Feynman diagrams of full QCD. However, as we are only interested in the soft and potential contributions, it is possible to apply step (i) to the diagrams constructed from the NRQCD Feynman rules. Equivalently, the Lagrangian of the effective theory can be employed for a perturbative calculation without explicit restrictions on the virtual momenta if dimensional regularization is used and the formal expressions derived from the Feynman rules of the effective theory are understood in the sense of the threshold expansion. The last remark is crucial because, in general, the naive use of the effective-theory Feynman rules and dimensional regularization leads to an incorrect result. Although this effect has not yet shown up in the QED bound-state calculations performed along these lines so far [28,29], it is important for our analysis.

As for the heavy-quark-antiquark potential, the contribution from the hard region is analytic in m_q^2 and starts, for dimensional reasons, at order $1/m_q^2$. By definition, the effect of the ultrasoft modes should not be included in the potential, so that we are left with the contributions of the soft and potential regions. In the effective-theory language, we study the reduction from NRQCD to PNRQCD and compute the effect of the soft and potential modes being integrated out.

For this purpose, it is enough to use the Lagrangian of NRQCD. Apart from the standard LO terms involving the gluon, ghost, quark, and antiquark fields in the Lagrangian, we need to also include the $1/m_q$ -suppressed terms originating from the covariant-derivative operator $D^2/(2m_q)$ acting on the quark and antiquark fields. Note that this operator also includes the quark kinetic-energy term $k^2/(2m_q)$. It can be either treated as a perturbation if k is soft or kept in the nonrelativistic quark propagator

$$S(k) = \frac{1}{k_0 - \mathbf{k}^2 / (2m_a) + i\varepsilon} \tag{1}$$

if *k* is potential. This issue is discussed below in more detail. By using the NRQCD Feynman rules, we immediately recover the well-known one-loop result,

$$\mathcal{V}_{1/m_q}^{(1)}(\boldsymbol{q}) = \frac{\pi^2 \alpha_s^2 C_F}{m_q |\boldsymbol{q}|} \left(\frac{C_F}{2} - C_A \right), \tag{2}$$

where $C_F = (N^2 - 1)/(2N)$ and $C_A = N$ are the eigenvalues of the quadratic Casimir operators of the fundamental and adjoint representations of the SU(N) color group, respectively. In coordinate space, Eq. (2) takes the form

$$\mathcal{V}_{1/m_q}^{(1)}(\mathbf{r}) = \frac{\alpha_s^2 C_F}{2m_o r^2} \left(\frac{C_F}{2} - C_A \right). \tag{3}$$

The Abelian part enters the one-loop expression because we use the on-shell potential. In fact, we shall need it for the renormalization of our two-loop result. In the standard QED analysis using the Coulomb gauge, it is removed by introducing in turn the tree-level operator

$$\mathcal{V}_{\text{off}} = \frac{\pi \alpha_s C_F}{m_a^2} \left(\frac{\mathbf{p}'^2 - \mathbf{p}^2}{\mathbf{q}^2} \right)^2, \tag{4}$$

where p and p' are the relative three-momenta of the quark and antiquark, respectively, and q=p'-p. This is a purely off-shell operator, which vanishes for on-shell quarks, when $p^2=p'^2=2m_qE$, where E is the quark energy counted from the threshold. By using the Coulomb equation of motion, it is straightforward to check that the matrix elements of the Abelian part of Eq. (2) and of Eq. (4) between Coulomb states are the same. The use of off-shell operators is advantageous in QED because it allows one to reduce the number of loops by means of the Coulomb equation of motion, as may be seen by comparing Eqs. (2) and (4). However, we use the on-shell formulation and the general covariant gauge, which is more suitable for multiloop-QCD calculations.

The calculation of the non-Abelian part of Eq. (2) can be greatly simplified by observing that the propagator of Eq. (1) can be expanded in $1/m_q$ not only in the soft region, but also in the potential one, which is related to the contribution of its pole to the integral over k_0 . This is possible because this part is free of Coulomb singularities. In fact, performing the expansion, one recovers the familiar generalized functions of k_0 , $\delta^{(n)}(k_0)$. By contrast, the Abelian part suffers from the

Coulomb singularity corresponding to the iterations of the off-shell operator of Eq. (4) and the Coulomb potential. Note that the iterations of the operators included in the effective Hamiltonian are taken into account by solving the Schrödinger equation perturbatively around the Coulomb solution. However, since we do not include the off-shell operator of Eq. (4) in the effective Hamiltonian, we must include the result of its iterations directly in the potential. After expanding the quark propagator, this produces ill-defined products like $1/[(k_0+i\varepsilon)^m(k_0-i\varepsilon)^n]$, which must be avoided by resorting to the unexpanded version of Eq. (1) in the potential region.

The structure of the expansion remains intact at two loops, and our final result reads

$$\mathcal{V}_{1/m_q}^{(2)}(\mathbf{q}) = \frac{\pi \alpha_s^3 C_F}{m_q |\mathbf{q}|} \left\{ \left[\frac{(C_F - 2C_A)\beta_0}{4} - \frac{4(2C_F + C_A)C_A}{3} \right] \ln \frac{\mu^2}{\mathbf{q}^2} + b_2 \right\},$$
 (5)

where

$$b_{2} = \left(\frac{65}{18} - \frac{8}{3}\ln 2\right) C_{F}C_{A} - \frac{2}{9}C_{F}T_{F}n_{I}$$
$$-\left(\frac{101}{36} + \frac{4}{3}\ln 2\right) C_{A}^{2} + \frac{49}{36}C_{A}T_{F}n_{I}, \tag{6}$$

 $\beta_0 = 11C_A/3 - 4T_F n_l/3$ is the one-loop coefficient of the QCD β function, $T_F = 1/2$ is the index of the fundamental representation, and n_l is the number of light-quark flavors. For N=3 and $n_l=3$, 4, and 5, we have $b_2 \approx -20.836$, -18.943, and -17.049, respectively. In Eq. (5), we omitted the purely Abelian part, since the QED potential to this order is known in the off-shell form. The result evaluated in dimensional regularization may be found, for example, in Ref. [28]. In coordinate space, Eq. (5) becomes

$$\mathcal{V}_{1/m_q}^{(2)}(\mathbf{r}) = \frac{\alpha_s^3 C_F}{2\pi m_q r^2} \left\{ \left[\frac{(C_F - 2C_A)\beta_0}{4} - \frac{4(2C_F + C_A)C_A}{3} \right] \ln(\tilde{\mu}^2 r^2) + b_2 \right\}, \quad (7)$$

where $\tilde{\mu} = e^{\gamma_E} \mu$, with γ_E being Euler's constant. In our two-loop analysis, we again used the expanded form of Eq. (1) in the calculation of the maximal non-Abelian structures, $C_F C_A^2$ and $C_F C_A T_F n_I$, while it was necessary to keep Eq. (1) unexpanded for the treatment of the Coulomb singularities associated with the $C_F^2 C_A$ and $C_F^2 T_F n_I$ terms.

We performed a number of nontrivial checks for our analysis. (i) We worked in the general covariant gauge and verified that the gauge parameter cancels in our final result. (ii) The two-loop expression from which Eq. (5) is obtained contains both ultraviolet (UV) and infrared (IR) divergences. The UV ones were removed in Eq. (5) by the renormalization of α_s in the one-loop result of Eq. (2), which we per-

formed in the modified minimal-subtraction (MS) scheme. By the same token, the renormalization group logarithms proportional to β_0 in Eq. (5) compensate the μ dependence of Eq. (2). On the other hand, the IR divergences are canceled by the UV ones of the ultrasoft contribution [22] leaving a finite, μ -independent result for the spectrum. (iii) To test our program we also recalculated the two-loop correction to the static heavy-quark—antiquark potential and found agreement with Ref. [8]. This is useful in its own right, since Ref. [8] disagrees with the original result of Ref. [7].

In Eq. (5), the IR divergences were subtracted according to the MS prescription. For consistency, the same prescription must be used for the calculation of the ultrasoft contribution. We should mention that, in Ref. [22], a slightly different regularization/subtraction scheme was used. There, only the divergent ultrasoft integrals were dimensionally regularized, while the remaining convergent integrals over the potential region were solved in four dimensions, despite overall factors containing poles in ϵ . This means that an additional matching term must be added to the result of Ref. [22] so as to convert it into the conventional MS scheme used in the present paper. In particular, this term contains logarithms that exactly cancel one half of the IR logarithms in Eq. (5). The remaining IR μ dependence is canceled by the ultrasoft logarithms, involving the scale E instead of |q|, that result in the logarithmically enhanced corrections to the bound-state parameters [23].

Our result is very relevant for heavy-quarkonium spectroscopy. In fact, the complete N³LO analysis of the spectrum includes three basic ingredients: (i) the leading retardation effect; (ii) the multiple iterations of the low-order corrections to the potential; and (iii) the N³LO potential, which includes $\mathcal{O}(v^2\alpha_s)$, $\mathcal{O}(v\alpha_s^2)$, and $\mathcal{O}(\alpha_s^3)$ terms. The retardation effect related to the radiation and absorption of ultrasoft gluons, which results in the OCD analogue of the well-known Bethe logarithms, was analyzed in Ref. [22]. The corrections to the spectrum due to multiple iterations of the low-order corrections to the potential can be obtained using the technique of Ref. [18]. So far, only the one-loop $\mathcal{O}(v^2\alpha_s)$ part of the N³LO potential has been available [4–6]. Now that also the $\mathcal{O}(v \alpha_s^2)$ part is known, only the three-loop $\mathcal{O}(\alpha_s^3)$ correction to the static potential remains to be computed. Although the latter analysis is tedious from the technical point of view, it is rather straightforward from the conceptual one. We believe that the calculation of the twoloop $\mathcal{O}(v \alpha_s^2)$ term performed here is probably most involved conceptually in the sense that the full power of modern technology in effective field theory is required.

Our result is also very important in view of the recent analysis of Ref. [24]. There, it was conjectured that the perturbation theory for the normalization of the $t\bar{t}$ production cross section at threshold can be essentially improved by reordering the corrections and converting the standard perturbative expansion in powers of α_s and v around the Coulomb solution to a logarithmic one, which sums up the infinite number of logarithmically enhanced terms in each order by solving the nonrelativistic renormalization group equation [30]. This conjecture is supported by an analysis involving

the known part of the NNLO anomalous dimensions, which is, however, incomplete. The remaining part of the next-to-next-to-leading logarithmic (NNLL) corrections can be obtained using an approach [29] developed for the calculation of the subleading logarithmic corrections in the spectrum of positronium bound states. The most nontrivial ingredient necessary to complete the calculation of the NNLO anomalous dimensions is the two-loop result given in the present paper. Note that the QED analysis [29] implies that the missed terms are likely to be important. In fact, their Abelian counterparts are responsible for the third-order subleading logarithmic corrections to the positronium spectrum [29]. If the complete analysis supported the conjecture of Ref. [24], this would be crucial for the precision study of Higgs-boson-induced effects in $t\bar{t}$ threshold production.

- Finally, we would like to mention that there is an interesting connection [31] between the perturbative $\alpha_s^n/(m_q r^2)$ potential and the nonperturbative calculation of the heavy-quark-antiquark potential in lattice QCD [32].
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